Universal Quantification

Let P(x) be a predicate (propositional function).

Universally quantified sentence:

For all x in the universe of discourse P(x) is true.

Using the universal quantifier $\forall$:

$\forall x\, P(x)$ "for all x P(x)" or "for every x P(x)"

(Note: $\forall x\, P(x)$ is either true or false, so it is a proposition, not a propositional function.)

Universal Quantification

Example: Let the universe of discourse be all people

$S(x)$: x is a UMBC student.

$G(x)$: x is a genius.

What does $\forall x\ (S(x) \rightarrow G(x))$ mean ?

"If x is a UMBC student, then x is a genius." or

"All UMBC students are geniuses."

If the universe of discourse is all UMBC students, then the same statement can be written as

$\forall x\ G(x)$

Existential Quantification

Existentially quantified sentence:

There exists an x in the universe of discourse for which P(x) is true.

Using the existential quantifier ∃:

$\exists x\, P(x)$ “There is an x such that P(x).”

“There is at least one x such that P(x).”

(Note: $\exists x\, P(x)$ is either true or false, so it is a proposition, but no propositional function.)

Existential Quantification

Example:

P(x): x is a UMBC professor.

G(x): x is a genius.

What does $\exists x\ (P(x) \wedge G(x))$ mean ?

"There is an x such that x is a UMBC professor and x is a genius."

or

"At least one UMBC professor is a genius."

Quantification

Another example:

Let the universe of discourse be the real numbers.

What does $\forall x \exists y\ (x + y = 320)$ mean ?

"For every x there exists a y so that x + y = 320."

Is it true? yes

Is it true for the natural numbers? no

Disproof by Counterexample

A counterexample to $\forall x\ P(x)$ is an object c so that P(c) is false.

Statements such as $\forall x\ (P(x) \rightarrow Q(x))$ can be disproved by simply providing a counterexample.

Statement: "All birds can fly."

Disproved by counterexample: Penguin.

Negation

$\neg(\forall x\ P(x))$ is logically equivalent to $\exists x\ (\neg P(x))$.

$\neg(\exists x\ P(x))$ is logically equivalent to $\forall x\ (\neg P(x))$.

See Table 2 in Section 1.3.

This is de Morgan's law for quantifiers

Negation

Examples

Not all roses are red

$\neg \forall x\ (Rose(x) \rightarrow Red(x))$

$\exists x\ (Rose(x) \wedge \neg Red(x))$

Nobody is perfect

$\neg \exists x\ (Person(x) \wedge Perfect(x))$

$\forall x\ (Person(x) \rightarrow \neg Perfect(x))$

Nested Quantifier

- A predicate can have more than one variables.
  - $S(x, y, z)$: z is the sum of x and y
  - $F(x, y)$: x and y are friends
- We can quantify individual variables in different ways
  - $\forall x, y, z\ (S(x, y, z) \rightarrow (x <= z \land y <= z))$
  - $\exists x\ \forall y\ \forall z\ (F(x, y) \land F(x, z) \land (y\ != z) \rightarrow \neg F(y, z)$

Nested Quantifier

Exercise: translate the following English sentence into logical expression

"There is a rational number in between every pair of distinct rational numbers"

Use predicate $Q(x)$, which is true when x is a rational number

$\forall x,y\ (Q(x) \land Q(y) \land (x < y) \rightarrow$
$\exists u\ (Q(u) \land (x < u) \land (u < y)))$

Summary, Sections 1.3, 1.4

- Propositional functions (predicates)
- Universal and existential quantifiers, and the duality of the two
- When predicates become propositions
  - All of its variables are instantiated
  - All of its variables are quantified
- Nested quantifiers
  - Quantifiers with negation
- Logical expressions formed by predicates, operators, and quantifiers

Let's proceed to…

Mathematical Reasoning

Mathematical Reasoning

We need **mathematical reasoning** to

- determine whether a mathematical argument is correct or incorrect and
- construct mathematical arguments.

Mathematical reasoning is not only important for conducting **proofs** and **program verification**, but also for **artificial intelligence** systems (drawing logical inferences from knowledge and facts).

We focus on **deductive** proofs

Terminology

An **axiom** is a basic assumption about mathematical structure that needs no proof.

- Things known to be true (facts or proven theorems)
- Things believed to be true but cannot be proved

We can use a **proof** to demonstrate that a particular statement is true. A proof consists of a sequence of statements that form an argument.

The steps that connect the statements in such a sequence are the **rules of inference**.

Cases of incorrect reasoning are called **fallacies**.

Terminology

A **theorem** is a statement that can be shown to be true.

A **lemma** is a simple theorem used as an intermediate result in the proof of another theorem.

A **corollary** is a proposition that follows directly from a theorem that has been proved.

A **conjecture** is a statement whose truth value is unknown. Once it is proven, it becomes a theorem.

Proofs

A **theorem** often has two parts
- Conditions (premises, hypotheses)
- conclusion

A **correct (deductive) proof** is to establish that
- If the conditions are true then the conclusion is true
- I.e., Conditions $\rightarrow$ conclusion is a tautology

Often there are missing pieces between conditions and conclusion. Fill it by an argument
- Using conditions and axioms
- Statements in the argument connected by proper rules of inference